Math 255A' Lecture 1 Notes

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September 27, 2019

1 Hilbert Space Review

1.1 Inner products

In functional analysis, we need to use a field with a topological structure. In this course, we will use the fields $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Definition 1.1. Let H be a vector space over $\mathbb{F}$. A **semi-inner product** $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{F}$ is a function such that

1. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$
2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
3. $\langle x, x \rangle \geq 0$.

This is an **inner product** if $\langle x, x \rangle = 0 \implies x = 0$.[1]

Example 1.1. $\mathbb{F}^n$ has the inner product $\langle x, y \rangle = \sum_{i=1}^n x_i \overline{y}_i$.

Example 1.2. $\mathbb{F}^\infty = \{(x_i)_{i=1}^\infty \in \mathbb{F}^{\mathbb{N}} : x_i = 0 \text{ for all sufficiently large } i\}$ has the inner product $\langle x, y \rangle = \sum_{i=1}^\infty x_i \overline{y}_i$.

Example 1.3. $L^2_{\mathbb{F}}(\mu) = \{f : X \to \mathbb{F} : f \text{ measurable}, \int |f|^2 \, d\mu < \infty\}$ has the inner product $\langle f, g \rangle = \int f \overline{g} \, d\mu$.

1.2 Norm and metric structure

Theorem 1.1 (Cauchy-Bunyakowski-Schwarz inequality). *Any semi-inner product satisfies*

$$|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \sqrt{\langle y, y \rangle}.$$

Corollary 1.1. *If we set* $\|x\| := \sqrt{\langle x, x \rangle}$*, then*

[1] This is sometimes referred to as the inequality being "coercive."

- $\|x+y\| \leq \|x\| + \|y\|$
- $\|\lambda x\| = |\lambda| \cdot \|x\| \qquad \forall \lambda \in \mathbb{F}, x \in H.$

Definition 1.2. $\|\cdot\|$ is called the **(semi-) norm** associated to the (semi-) inner product.

Proposition 1.1 (Polar identity)**.**

$$\|x+y\|^2 = \|x\|^2 + 2\operatorname{Re}(\langle x, y\rangle) + \|y\|^2$$

Remark 1.1. We get the imaginary part, too, because

$$\operatorname{Re}\langle -ix, y\rangle = \operatorname{Re}(-i\langle x, y\rangle) = \operatorname{Im}\langle x, y\rangle .$$

Definition 1.3. The **associated metric** to an inner product is $d(x,y) := \|x-y\|$.

Definition 1.4. A **Hilbert space** is an inner product space which is complete with respect to this metric.

Example 1.4. $\mathbb{F}^n$ is a Hilbert space.

Example 1.5. $\mathbb{F}^\infty$ is not complete, so it is not a Hilbert space.

Example 1.6. $L^2(\mu)$ is a Hilbert space.

Proposition 1.2. *If $(H, \langle\cdot,\cdot\rangle)$ is an inner product space, then there is a Hilbert space $(H', \langle\cdot,\cdot\rangle')$ such that*

- *$H \subseteq H'$, and H is dense,*
- *$\langle\cdot,\cdot\rangle'|_{H\times H} = \langle\cdot,\cdot\rangle$.*

The space H' is called the **completion** of H.

Example 1.7. The completion of $\mathbb{F}^\infty$ is $\ell^2 = \{(x_i)_{i=1}^\infty \in \mathbb{F}^\mathbb{N} : \sum_{i=1}^\infty |x_i|^2 < \infty\}$ with the inner product $\langle x, y\rangle = \sum_{i=1}^\infty x_i \overline{y}_i$. This is also $L^2(m)$, where m is counting measure on $\mathbb{N}$.

Example 1.8. Let $G \subseteq \mathbb{C}$ be open. Then the **Bergman space** $L_a^2(G)$, the set of L^2 functions that are analytic in G, is a Hilbert space.

1.3 Orthogonality

Definition 1.5. Elements $x, y \in H$ are **orthogonal** (denoted $x \perp y$) if $\langle x, y\rangle = 0$. If $A, B \subseteq H$, we say $A \perp B$ if $x \perp y$ for all $(x,y) \in A \times B$.

Theorem 1.2 (Pythagorean identity)**.** *Let H be a semi-inner product space, and let $x_n \in H$ be such that $x_i \perp x_j$ for all $i \neq j$. Then*

$$\|x_1 + \cdots + x_n\|^2 = \|x_1\|^2 + \cdots + \|x_n\|^2.$$

Corollary 1.2 (Parallelogram law)**.** *For any $x, y \in H$,*

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Definition 1.6. $A \subseteq H$ is **convex** if whenever $x, y \in A$, $tx + (1-t)y \in A$ for all $t \in [0,1]$.

Proposition 1.3. *Let H be a Hilbert space, let $h \in H$, and let $K \subseteq H$ be nonempty, closed, and convex. Then there is a unique $k \in K$ such that $\|h-k\| \leq \|h-k'\|$ for all $k' \in K$.*

Corollary 1.3. *This holds if K is a closed subspace of H.*

Theorem 1.3. *If M is a closed subspace of a Hilbert space and $h \in H$, then $f \in M$ is the closest point to h iff $f \in M$ and $h - f \perp M$.*

Definition 1.7. If $A \subseteq H$, the **orthogonal complement** of A is $A^\perp = \{h \in H : h \perp A\}$.

Remark 1.2. For any A, $A^\perp$ is a closed, linear subspace.[2]

Theorem 1.4. *Let $M \subseteq H$, $h \in H$, and let Ph be the closest point in M to h. Then*

1. $P(ah + h') = aPh + Ph'$
2. $\|Ph\| \leq \|h\|$
3. $P^2 h = Ph$
4. $\ker P = M^\perp$, *and* $\operatorname{im} P = M$.

Definition 1.8. $P = P_M$ is called the **orthogonal projection** onto M.

Corollary 1.4. $(A^\perp)^\perp = \overline{\operatorname{span}} A$.

Corollary 1.5. *If Y is a linear subspace of H, then Y is dense in H if and only if $Y^\perp = \{0\}$.*

[2]You could put in a picture of a rabbit, and $A^\perp$ would be a closed subspace.